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Analysis of electromagnetic scattering from an overfilled cavity in the ground plane

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In memoriam: Dr. William D. Wood Jr. (1963–2004)

Abstract

In this paper, we consider the scattering of the time-harmonic plane wave by a protruding cavity embedded in the PEC ground plane. An artificial boundary condition is introduced on a semicircle enclosing the cavity that couples the fields from the infinite exterior domain to those inside. Variational formulations for the TM and TE polarizations are derived and existence and uniqueness of weak solutions are established. Finite element error analysis is also performed. Numerical experiments demonstrate the efficiency and accuracy of the method.

1. Introduction

The analysis of the electromagnetic scattering properties of cavities in a conducting ground plane is of high interest to the engineering community. Applications include the design of cavity-backed conformal antennas for civil and military use, and the characterization of radar cross-section (RCS) of vehicles with grooves.

Time-harmonic analysis of cavity-backed apertures with penetrable material filling the cavity interior has been examined by numerous researchers in the engineering community. They include high and low frequency methods [12,16,6], the method of moments [20,21], and hybrid methods [15,18,14,13,17]. Mathematical treatment of scattering problems involving cavities can be found in [5,3,4,19]. It is a common assumption that the cavity opening coincides with the aperture on an infinite ground plane, and hence simplifying the modelling of the exterior (to the cavity) domain. This limits the application of these methods since many cavity openings are not planar. This paper aims to develop a solid mathematical technique that is capable of characterizing the scattering by overfilled cavities in the frequency domain.

In particular, we seek to determine the fields scattered by the protruding cavity upon a given incident wave. Our method decomposes the entire solution domain to two sub-domains via an artificial semicircle enclosing the cavity: the infinite upper half plane over the perfect electrically conducting (PEC) ground plane exterior to

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the semicircle, and the cavity plus the interior region. The problem is solved exactly in the infinite sub-domain, while the other is solved using finite elements. The two regions are coupled over the semicircle via the introduction of a boundary operator exploiting the field continuity over material interfaces. This is an important departure from the author's previous work, in that the fields above the aperture are not represented using a half-space Green's function. Rather, a modal representation is used to express the fields in the upper half space less a semicircle (2D) or hemisphere (3D) centered on the aperture and enclosing the inhomogeneities above it. In this way, cavity-backed antennas with dielectric lenses above the ground plane can be rigorously analyzed.

The idea of using a pseudo-differential operator to reduce the infinite computational domain to a finite one goes back at least to the classic work of Engquist and Majda [8]. A sequence of boundary conditions was developed in [1] which provided increasingly accurate approximations to an elliptic problem of infinite domain. Various expansions for symmetric geometries are used to generate Dirichlet to Newmann boundary operators in [9–11]. For more detailed analysis of higher order boundary conditions for wave equations the reader is referred to a recent paper by Diaz and Joly [7] and references therein.

2. Problem setting

Let $\Omega \subset \mathbb{R}^2$ be the cross-section of a z-invariant cavity (or trough) in the infinite ground plane such that its fillings of relative permittivity $\varepsilon_r \geqslant 1$ protrude above the ground plane. Denote S as the cavity wall, Γ the cavity aperture so that $\partial\Omega = S \cup \Gamma$. The infinite ground plane excluding the cavity opening is denoted as Γ_{ext}, the infinite homogenous region above the cavity as $\mathscr{U} = \mathbb{R}^2_+ \setminus \Omega$. Furthermore, let $\mathscr{B}_R$ be a semicircle of radius R large enough to completely enclose the overfilled portion of the cavity. We denote the region bounded by $\mathscr{B}_R$ and the cavity wall S as Ω_R. Hence, this region Ω_R consists of the cavity and the homogeneous part between $\mathscr{B}_R$ and Γ (see Fig. 1). Let $\mathscr{U}_R$ be the homogeneous region outside of Ω_R, that is, $U_R = \{(r,\theta) : r > R, 0 < \theta < \pi\}$.

Given the incident electromagnetic wave $(\boldsymbol{E}^{\text{i}}, \boldsymbol{H}^{\text{i}})$ impinging on the protruding cavity, we wish to determine the resulting scattered fields $(\boldsymbol{E}^{\text{s}}, \boldsymbol{H}^{\text{s}})$.

Due to the uniformity in the z-axis, the scattering problem can be decomposed into two fundamental polarizations: transverse magnetic (TM) and transverse electric (TE). Its solution then can be expressed as a linear combination of the solutions to TM and TE problems.

In the TM polarization, the magnetic field $\boldsymbol{H}$ is transverse to the z-axis so that $\boldsymbol{E}$ and $\boldsymbol{H}$ are of the form

$$\boldsymbol{E} = (0, 0, E_z), \qquad \boldsymbol{H} = (H_x, H_y, 0).$$

In this case, the nonzero component of the total field satisfies the following problem:

$$\text{(TM)} \quad \begin{cases} \Delta E_z + k^2 \varepsilon_r E_z = 0 & \text{in } \Omega \cup \mathscr{U}, \\ E_z = 0 & \text{on } S \cup \Gamma_{\text{ext}}, \end{cases}$$

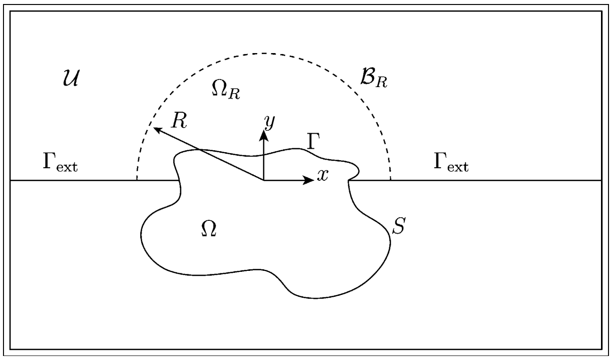

Fig. 1. Cavity geometry, showing cavity interior Ω and conducting boundary $S \cup \Gamma$. The conducting ground plane less the aperture is denoted Γ_{ext}, and $\mathscr{B}_R$ is an origin-centered semicircle surrounded by free space.

where $\varepsilon_r = \varepsilon/\varepsilon_0$ is the relative electric permittivity, and k is the free space wave number. Adapting the e^{-jwt} convention, we assume $\mathrm{Re}\,\varepsilon(x) \geqslant \alpha > 0$, $\mathrm{Im}\,\varepsilon(x) \leqslant 0$ and $\varepsilon(x) \in L^\infty(\Omega)$. The homogeneous region $\mathscr{U}$ above the protruding cavity is assumed to be air and hence its permittivity is $\varepsilon_r = 1$. In $\mathscr{U}$, the total field can be decomposed as $E_z = E_z^i + E_z^r + E_z^s$ where E_z^i is the incident field, E_z^r the reflected field, and E_z^s the scattered field. The reflected field exists due to the presence of the infinite ground plane. The incident and reflected electric fields satisfy

$$E_z^i + E_z^r = 0 \quad \text{on } \Gamma_{\text{ext}} \subset \{(x, y) : y = 0\}.$$

The scattered field E_z^s is governed by the following:

$$(\mathrm{TM}^s) \quad \begin{cases} \Delta E_z^s + k^2 E_z^s = 0 & \text{in } \mathscr{U}, \\ E_z^s = E_z - E_z^i - E_z^r & \text{on } \Gamma, \\ E_z^s = 0 & \text{on } \Gamma_{\text{ext}} \end{cases}$$

and the radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial E_z^s}{\partial r} + \mathrm{i} k E_z^s \right) = 0. \tag{2.1}$$

The components of $\boldsymbol{H}$ can be obtained in terms of E_z and its partial derivatives by using Maxwell's equations.

Similarly, in the TE polarization, the electric field $\boldsymbol{E}$ is transverse to the z-axis and hence,

$$\boldsymbol{E} = (E_x, E_y, 0), \qquad \boldsymbol{H} = (0, 0, H_z).$$

The nonzero component of the total magnetic field, also denoted by $\boldsymbol{H}$, satisfies the following problem:

$$(\mathrm{TE}) \quad \begin{cases} \nabla \cdot \left(\frac{1}{\varepsilon_r} \nabla H_z \right) + k^2 H_z = 0 & \text{in } \Omega \cup \mathscr{U}, \\ \frac{\partial H_z}{\partial n} = 0 & \text{on } S \cup \Gamma_{\text{ext}}. \end{cases}$$

In $\mathscr{U}$, the total magnetic field can be decomposed into $H_z = H_z^i + H_z^r + H_z^s$, where

$$\frac{\partial H_z^i}{\partial y} + \frac{\partial H_z^r}{\partial y} = 0 \quad \text{on } \{(x, y) : y = 0\}.$$

The scattered field solves

$$(\mathrm{TE}^s) \quad \begin{cases} \Delta H_z^s + k^2 H_z^s = 0 & \text{in } \mathscr{U}, \\ \frac{\partial H_z^s}{\partial n} = 0 & \text{on } \Gamma_{\text{ext}}, \end{cases}$$

where $\frac{\partial}{\partial n}$ is the normal derivative on Γ. The scattered magnetic field also satisfies the same radiation condition defined in (2.1). Again, the components of $\boldsymbol{E}$ can be obtained in terms of H_z and its partial derivatives by using Maxwell's equations.

3. TM polarization

In what follows, we shall denote $u = E_z$ for simplicity. The scattered field u^s satisfies the following *exterior problem*:

$$\begin{cases} \Delta u^s + k^2 u^s = 0 & \text{in } \mathscr{U}_R, \\ u^s(R, \theta) = g(\theta) & \text{on } \mathscr{B}_R, \\ u^s = 0 & \text{on } \Gamma_{\text{ext}} \end{cases} \tag{3.1}$$

and the radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} + \mathrm{i} k u^s \right) = 0. \tag{3.2}$$

In polar coordinates, the Helmholtz equation in (3.1) becomes

$$\frac{\partial^2 u^s}{\partial r^2} + \frac{1}{r}\frac{\partial u^s}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u^s}{\partial \theta^2} + k^2 u^s = 0. \tag{3.3}$$

By writing

$$u^s(r,\theta) = \sum_{n=0}^{\infty} u_n^s(r)(a_n \cos n\theta + b_n \sin n\theta)$$

and substituting it in (3.3) we get

$$\frac{\partial^2 u_n^s}{\partial r^2} + \frac{1}{r}\frac{\partial u_n^s}{\partial r} + \left(k^2 - \frac{n^2}{r^2}\right)u_n^s = 0.$$

The radiation condition then gives

$$u^s(r,\theta) = \sum_{n=0}^{\infty} H_n^{(2)}(kr)(a_n \cos n\theta + b_n \sin n\theta).$$

The PEC boundary conditions

$$0 = u^s(r,0) = \sum_{n=0}^{\infty} a_n H_n^{(2)}(kr), \quad r \geqslant R,$$
$$0 = u^s(r,\pi) = \sum_{n=0}^{\infty} (-1) a_n H_n^{(2)}(kr), \quad r \geqslant R,$$

further imply that $a_n = 0$ for all n. Hence,

$$u^s(r,\theta) = \sum_{n=1}^{\infty} b_n H_n^{(2)}(kr) \sin n\theta. \tag{3.4}$$

Letting $r = R$ and imploring the orthogonality of the sine functions yield

$$b_n = \frac{2}{\pi H_n^{(2)}(kR)} \int_0^{\pi} g(\theta) \sin n\theta \, \mathrm{d}\theta. \tag{3.5}$$

By taking the partial derivative of u^s with respect to r, we get

$$\frac{\partial u^s}{\partial r} = \frac{2k}{\pi} \sum_{n=1}^{\infty} \frac{H_n^{(2)'}(kr)}{H_n^{(2)}(kR)} \sin n\theta \int_0^{\pi} g(\theta) \sin n\theta \, \mathrm{d}\theta \equiv Tg(\theta) \tag{3.6}$$

for all $g \in H^{1/2}(\mathscr{B}_R)$.

The Sobolev spaces $H^{1/2}(\mathscr{B}_R)$ and $H^{-1/2}(\mathscr{B}_R)$ are defined as follows:

$$H^{1/2}(\mathscr{B}_R) = \left\{ \phi : \sum_{m=0}^{\infty} \sqrt{1+m^2} |\phi_m|^2 < \infty \right\}, \tag{3.7}$$

$$H^{-1/2}(\mathscr{B}_R) = \left\{ \phi : \sum_{m=0}^{\infty} \frac{1}{\sqrt{1+m^2}} |\phi_m|^2 < \infty \right\}, \tag{3.8}$$

where

$$\phi_m = \phi_m^c + \mathrm{i}\phi_m^s = \frac{2}{\pi} \int_0^{\pi} \phi(\theta) \mathrm{e}^{\mathrm{i}m\theta} \, \mathrm{d}\theta.$$

Let T_R be the restriction of T to $\mathscr{B}_R$ so that $T_R : H^{1/2}(\mathscr{B}_R) \to H^{-1/2}(\mathscr{B}_R)$ can be defined by

$$T_R w(\theta) = \frac{2k}{\pi} \sum_{n=1}^{\infty} \frac{H_n^{(2)'}(kR)}{H_n^{(2)}(kR)} \sin n\theta \int_0^{\pi} w(\theta) \sin n\theta \, \mathrm{d}\theta = k \sum_{n=1}^{\infty} \frac{H_n^{(2)'}(kR)}{H_n^{(2)}(kR)} \sin n\theta w_n^s$$

for all $w \in H^{1/2}(\mathscr{B}_R)$.

Lemma 3.1. *The operator* $T_R : H^{\frac{1}{2}}(\mathscr{B}_R) \to H^{-\frac{1}{2}}(\mathscr{B}_R)$ *is continuous.*

Proof. We denote

$$B_n = \frac{H_n'^{(2)}(kR)}{H_n^{(2)}(kR)}. \tag{3.9}$$

Then, see [2],

$$|B_n| = \left|\frac{K_n'(\mathrm{i}kR)}{K_n(\mathrm{i}kR)}\right|.$$

By the formula

$$zK_n'(z) = -nK_n(z) - zK_{n-1}(z),$$

we deduce

$$\left|\frac{K_n'(\mathrm{i}kR)}{K_n(\mathrm{i}kR)}\right| = \left|\frac{n}{\mathrm{i}x} + \frac{K_{n-1}(\mathrm{i}x)}{K_n(ix)}\right| \leqslant \frac{n}{x} + 1,$$

where $x = kR$. Hence $|B_n| \leqslant C\sqrt{n^2+1}$ for some $C > 0$. By definition, we have

$$\begin{aligned}
|\langle T_R w, \psi\rangle| &= \left|\int_{B_R} k \sum_1^\infty B_n \sin n\theta w_n^{\mathrm{s}} \bar{\psi}\,\mathrm{d}l\right| = \left|\int_0^\pi k \sum_1^\infty B_n \sin n\theta w_n^{\mathrm{s}} \bar{\psi}(\theta) R\,\mathrm{d}\theta\right| = \left|x \sum_1^\infty B_n w_n^{\mathrm{s}} \int_0^\pi \bar{\psi}(\theta) \sin n\theta\,\mathrm{d}\theta\right| \\
&= \left|\frac{x\pi}{2} \sum_1^\infty B_n w_n^{\mathrm{s}} \bar{\psi}_n^{\mathrm{s}}\right| \leqslant \frac{x\pi}{2} \sqrt{\sum_1^\infty |B_n||w_n^{\mathrm{s}}|^2} \sqrt{\sum_1^\infty |B_n||\psi_n^{\mathrm{s}}|^2} \\
&\leqslant C\left(\sum_1^\infty \sqrt{1+n^2}|w_n^{\mathrm{s}}|^2\right)^{1/2} \left(\sum_1^\infty \sqrt{1+n^2}|\psi_n^{\mathrm{s}}|^2\right)^{1/2} = C\|w\|_{H^{1/2}(\mathscr{B}_R)} \cdot \|\psi\|_{H^{1/2}(\mathscr{B}_R)}.
\end{aligned}$$

Thus we have

$$\|T_R w\|_{H^{-1/2}(\mathscr{B}_R)} = \sup_{\psi \in H^{1/2}(\mathscr{B}_R)} \frac{|\langle T_R w, \psi\rangle|}{\|\psi\|_{H^{1/2}(\mathscr{B}_R)}} \leqslant C\|w\|_{H^{1/2}(\mathscr{B}_R)}. \qquad \square$$

On the semicircle $\mathscr{B}_R$, where $r = R$, the normal derivative of the total electric field can be decomposed as the following:

$$\frac{\partial u}{\partial r} = \frac{\partial u^{\mathrm{i}}}{\partial r} + \frac{\partial u^{\mathrm{r}}}{\partial r} + \frac{\partial u^{\mathrm{s}}}{\partial r} = \frac{\partial u^{\mathrm{i}}}{\partial r} + \frac{\partial u^{\mathrm{r}}}{\partial r} + T_R(u^{\mathrm{s}}) \equiv f(\theta) + T_R(u) - T_R(u^{\mathrm{i}} + u^{\mathrm{r}}).$$

By field continuity we can now reduce the problem (TM) defined in the infinite domain $\Omega \cup \mathscr{U}$ to the following *interior problem*:

$$\begin{cases} \Delta u + k^2 \varepsilon_{\mathrm{r}} u = 0 & \text{in } \Omega_R, \\ \frac{\partial u}{\partial r} - T_R(u) = f(\theta) - T_R(u^{\mathrm{i}} + u^{\mathrm{r}}) & \text{on } \mathscr{B}_R. \end{cases} \tag{3.10}$$

In what follows, (3.10) will be solved by a variational method.

3.1. Variational formulation

Define the sub-space V of $L^2(\Omega_R)$ by

$$V = \{v \in H^1(\Omega_R) : v|_S = 0\}$$

equipped with the H^1-norm

$$\|u\|_V = \|u\|_{H^1(\Omega_R)}.$$

The variational formulation of (3.10) is to find $u \in V$ such that

$$b_{\mathrm{TM}}(u,v) = F(v) \quad \text{for all } v \in V, \tag{3.11}$$

where

$$\begin{aligned} b_{\mathrm{TM}}(u,v) &= \int_{\Omega_R} (\nabla u \cdot \nabla v - k^2 \varepsilon_{\mathrm{r}} uv)\,\mathrm{d}x\,\mathrm{d}y - \int_{\mathscr{B}_R} T_R(u) v\,\mathrm{d}l \\ &= \int_{\Omega_R} (\nabla u \cdot \nabla v - k^2 \varepsilon_{\mathrm{r}} uv)\,\mathrm{d}x\,\mathrm{d}y - \int_0^{\pi} T_R(u) vR\,\mathrm{d}\theta, \end{aligned} \tag{3.12}$$

$$\begin{aligned} F(v) &= \int_{\mathscr{B}_R} [f(\theta) - T_R(u^{\mathrm{i}} + u^{\mathrm{r}})] v\,\mathrm{d}l \\ &= \int_0^{\pi} [f(\theta) - T_R(u^{\mathrm{i}} + u^{\mathrm{r}})] vR\,\mathrm{d}\theta. \end{aligned} \tag{3.13}$$

Theorem 3.2. *The variational problem (3.13) has a unique solution $u \in V$.*

Proof. The proof consists of two parts. First we show

$$\mathrm{Re}\{b_{\mathrm{TM}}(u,u)\} \geqslant C_1 \|\nabla u\|^2_{L^2(\Omega)} - C_2 \|u\|^2_{L^2(\Omega)}$$

for some $C_1 > 0$, $C_2 > 0$. Secondly, we show the variational problem (3.13) can have at most one solution. Then existence of solution follows immediately from Fredholm alternative theorem. We observe that

$$\mathrm{Re}\, B_n = \frac{J_n J_{n+1} + Y_n Y_{n+1}}{J_n^2 + Y_n^2} = \frac{n}{x} - \frac{J_n J_{n+1} + Y_n Y_{n+1}}{J_n^2 + Y_n^2}, \tag{3.14}$$

$$\mathrm{Im}\, B_n = \frac{J_n' Y_n - J_n Y_n'}{J_n^2 + Y_n^2} = \frac{-W(J_n, Y_n)}{J_n^2 + Y_N^2} = -\frac{2/\pi x}{J_n^2 + Y_n^2}, \tag{3.15}$$

where $x = kR$ and B_n is as defined in (3.9). It can be shown [2], that $\mathrm{Re}\, B_n \leqslant 0\ \forall n \geqslant 0$ and $\forall kR > 0$, while it is clear that $\mathrm{Im}\, B_n \leqslant 0$. Hence we have

$$\begin{aligned} \mathrm{Re}\, b_{\mathrm{TM}}(u,u) &= \|\nabla u\|^2_{L^2_{\Omega_R}} - k^2 \int_{\Omega_R} \mathrm{Re}\, \varepsilon_{\mathrm{r}} |u|^2\,\mathrm{d}x\,\mathrm{d}y - \mathrm{Re}\,\langle T_R u, u\rangle \\ &= \|\nabla u\|^2_{L^2_{\Omega_R}} - k^2 \int_{\Omega_R} \mathrm{Re}\, \varepsilon_{\mathrm{r}} |u|^2\,\mathrm{d}x\,\mathrm{d}y - \frac{x\pi}{2} \sum_{n=1}^{\infty} \mathrm{Re}\, B_n (u_n^{\mathrm{s}})^2 \geqslant \|\nabla u\|^2_{L^2(\Omega_R)} - k^2 \int_{\Omega_R} \mathrm{Re}\, \varepsilon_{\mathrm{r}} |u|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\geqslant C_1 \|\nabla u\|^2_{L^2(\Omega_R)} - C_2 \|u\|_{L^2(\Omega_R)}. \end{aligned}$$

To prove uniqueness, assume u is a solution of (3.13) with $F(v) = 0$. We need only show that $u \equiv 0$. Indeed, $b_{\mathrm{TM}}(u,u) = 0$ implies

$$\mathrm{Im}\, b_{\mathrm{TM}}(u,u) = -k^2 \int_{\Omega_R} \mathrm{Im}\, \varepsilon_{\mathrm{r}} |u|^2\,\mathrm{d}x\,\mathrm{d}y - \mathrm{Im}\,\langle T_R u, u\rangle = 0.$$

Note that $\mathrm{Im}\, \varepsilon_{\mathrm{r}} \leqslant 0$, we deduce

$$k^2 \int_{\Omega_R} |\mathrm{Im}\, \varepsilon_{\mathrm{r}}| \cdot |u|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{\pi x}{2} \sum_{n=1}^{\infty} \frac{2}{\pi x} \frac{1}{J_n^2 + Y_n^2} (u_n^{\mathrm{s}})^2 = 0.$$

Hence, $u \equiv 0$. $\square$

3.2. TE polarization

As in the TM case, we denote $u = H_z$ for simplicity. On the PEC ground plane, Γ_{ext}, it is known that

$$\frac{\partial u^{\text{i}}}{\partial n} + \frac{\partial u^{\text{r}}}{\partial n} = 0.$$

In this case, the unit normal vector $\hat{n}$ is in the positive y-axis. The scattered field u^{s} satisfies the following *exterior problem*:

$$\begin{cases} \Delta u^{\text{s}} + k^2 u^{\text{s}} = 0 & \text{in } \mathscr{U}_R, \\ u^{\text{s}}(R,\theta) = h(\theta) & \text{on } \mathscr{B}_R, \\ \frac{\partial u^{\text{s}}}{\partial r} = 0 & \text{on } \Gamma_{\text{ext}}, \end{cases} \tag{3.16}$$

and the radiation condition as in (2.1).

We expand the scattered field u^{s} in $\mathscr{U}_R$ as

$$u^{\text{s}}(r,\theta) = 2a_0 H_0^{(2)}(kr) + \sum_{n=1}^{\infty} a_n H_n^{(2)}(kr)\cos n\theta, \quad r \geqslant R, \ \theta \in [0,\pi]. \tag{3.17}$$

Here the cosine series expansion is chosen because $\frac{\partial u^{\text{s}}}{\partial r}$ vanishes for $\theta = 0$, π and $r \geqslant R$. Letting $r = R$ and imploring the orthogonality of the cosine functions yield

$$a_n = \frac{2}{\pi H_n^{(2)}(kR)} \int_0^{\pi} h(\theta)\cos n\theta \, d\theta. \tag{3.18}$$

By taking the partial derivative of u^{s} with respect to r, we get

$$\frac{\partial u^{\text{s}}}{\partial r} = \frac{2k}{\pi} \sum_{n=0}^{\infty} \frac{H_n^{(2)'}(kr)}{H_n^{(2)}(kR)} \cos n\theta \int_0^{\pi} h(\theta)\cos n\theta \, d\theta.$$

On the semicircle $\mathscr{B}_R$, we denote

$$\frac{\partial u^{\text{s}}}{\partial r} = Sh(\theta),$$

where S is the boundary operator

$$Sw(\theta) = \frac{2k}{\pi} \sum_{0}^{\infty} \frac{H_n^{(2)'}(kR)}{H_n^{(2)}(kR)} \cos n\theta \int_0^{\pi} w(\theta)\cos n\theta \, d\theta,$$

for all $w \in H^{1/2}(\mathscr{B}_R)$.

As in the TM case, we have the following result for the operator S, whose proof is similar in nature and is omitted here for brevity.

Lemma 3.3. *The operator $S : H^{\frac{1}{2}}(\mathscr{B}_R) \to H^{-\frac{1}{2}}(\mathscr{B}_R)$ is continuous.*

Again, by the mapping S and continuity condition we reduce the problem (TE) defined in the infinite domain $\Omega \cup \mathscr{U}$ for the total magnetic field, u, to the following *interior problem*:

$$\begin{cases} \nabla \cdot (\varepsilon_{\text{r}}^{-1} \nabla u) + k^2 u = 0 & \text{in } \Omega_R, \\ \partial u/\partial r - S(u) = -S(u^{\text{i}} + u^{\text{r}}) + \frac{\partial u^{\text{i}}}{\partial r} + \frac{\partial u^{\text{r}}}{\partial r} & \text{on } \mathscr{B}_R, \\ \partial u/\partial n = 0 & \text{on } S. \end{cases} \tag{3.19}$$

3.3. Variational formulation

Define the variational space $W = H^1(\Omega_R)$. The variational formulation of (3.19) is to find $u \in W$ such that

$$b_{\text{TE}}(u,w) = G(w) \quad \text{for all } w \in W, \tag{3.20}$$

where

$$b_{\mathrm{TE}}(u,w)=\int_{\Omega_R}(\varepsilon_{\mathrm{r}}^{-1}\nabla u\cdot\nabla w-k^2uw)\,\mathrm{d}x\,\mathrm{d}y-\int_{\mathscr{B}_R}S(u)\varepsilon_{\mathrm{r}}^{-1}w\,\mathrm{d}l$$
$$=\int_{\Omega_R}(\varepsilon_{\mathrm{r}}^{-1}\nabla u\cdot\nabla w-k^2uw)\,\mathrm{d}x\,\mathrm{d}y-\int_0^{\pi}S(u)\varepsilon_{\mathrm{r}}^{-1}wR\,\mathrm{d}\theta,$$
$$G(w)=\int_{\mathscr{B}_R}\left[\frac{\partial u^{\mathrm{i}}}{\partial r}+\frac{\partial u^{\mathrm{r}}}{\partial r}-S(u^{\mathrm{i}}+u^{\mathrm{r}})\right]\varepsilon_{\mathrm{r}}^{-1}w\,\mathrm{d}l$$
$$=\int_0^{\pi}\left[\frac{\partial u^{\mathrm{i}}}{\partial r}+\frac{\partial u^{\mathrm{r}}}{\partial r}-S(u^{\mathrm{i}}+u^{\mathrm{r}})\right]\varepsilon_{\mathrm{r}}^{-1}wR\,\mathrm{d}\theta.$$

Theorem 3.4. *The variational problem (3.20) has a unique solution $u\in W$.*

The proof is similar in nature to the proof for the TM case. We highlight the following.

Proof. We observe that

$$\langle Sw,\psi\rangle=nt_{B_R}k\sum_1^{\infty}B_n\cos n\theta w_n^{\mathrm{c}}\bar{\psi}\,\mathrm{d}l=\int_0^{\pi}k\sum_1^{\infty}B_n\cos n\theta w_n^{\mathrm{c}}\bar{\psi}(\theta)R\,\mathrm{d}\theta=x\sum_1^{\infty}B_nw_n^{\mathrm{c}}\int_0^{\pi}\bar{\psi}(\theta)\cos n\theta\,\mathrm{d}\theta$$
$$=\frac{x\pi}{2}\sum_1^{\infty}B_nw_n^{\mathrm{c}}\bar{\psi}_n^{\mathrm{c}}.$$

Hence we have

$$\mathrm{Re}\,b_{\mathrm{TE}}(u,u)=\int_{\Omega_R}\mathrm{Re}\,\varepsilon_{\mathrm{r}}^{-1}|\nabla u|^2-k^2|u|^2\,\mathrm{d}x\,\mathrm{d}y-\mathrm{Re}\,\langle Su,u\rangle$$
$$=\int_{\Omega_R}\mathrm{Re}\,\varepsilon_{\mathrm{r}}^{-1}|\nabla u|^2-k^2|u|^2\,\mathrm{d}x\,\mathrm{d}y-\frac{x\pi}{2}\sum_{n=1}^{\infty}\mathrm{Re}\,B_n(u_n^{\mathrm{c}})^2=\int_{\Omega_R}\mathrm{Re}\,\varepsilon_{\mathrm{r}}^{-1}|\nabla u|^2-k^2|u|^2\,\mathrm{d}x\,\mathrm{d}y$$
$$\geqslant C_1\|\nabla u\|^2_{L^2(\Omega_R)}-C_2\|u\|_{L^2(\Omega_R)}.$$

The uniqueness proof is essentially the same as that in Theorem 3.2. □

4. Finite element analysis

The variational equations (3.11) and (3.20), of the TM and TE problems, respectively, can be numerically solved by using finite element methods. In this section, we discuss the convergence properties of the finite element solutions.

Let $\{V_h:0<h<1\}$ be a family of finite dimensional sub-spaces of V. The corresponding discrete problem of (3.13) is: given the incident field u^{i}, find $u_h\in V_h$ such that

$$b(u_h,v_h)=F(v_h)\quad\forall v_h\in V_h.\tag{4.1}$$

The well-posedness of (4.1) follows immediately from Theorem 3.4.

Theorem 4.1. *Let $u\in V$ be the unique weak solution to (3.13). Then for any given $\epsilon>0$, there exists $h_0=h_0(\epsilon)>0$ such that for $0<h<h_0$, any solution $u_h\in V_h$ to (4.1) satisfies*

$$\|u-u_h\|_{H^1(\Omega_R)}=\mathscr{O}(\epsilon),\tag{4.2}$$
$$\|u-u_h\|_{L^2(\Omega_R)}=\mathscr{O}(\epsilon^2).\tag{4.3}$$

Remark 4.2. For a proof of the above theorem, the reader is referred to [19] Theorem 1.10.

The corresponding discrete problem of (3.20) is: given the incident field u^{i}, find $u_h \in V_h$ such that

$$b(u_h, v_h) = G(v_h) \quad \forall v_h \in V_h. \tag{4.4}$$

The same error estimate results as in the TM case hold for the TE case. The proofs are similar in nature and are skipped here for brevity.

5. Numerical experiments

In this section we present some numerical results for the case of an overfilled cavity where the protruding portion is a semicircle of radius $R_\Omega = 0.6$ m and the interior semicircle is of radius 0.5 m. The problem geometry is depicted in the contour plots, Figs. 2 and 3. The cavity is illuminated by a 300 MHz wave. In particular, we assume $u^{\mathrm{i}} = \mathrm{e}^{x\alpha+y\beta}$ and $u^{\mathrm{r}} = -\mathrm{e}^{x\alpha-y\beta}$ for the TM polarization; $u^{\mathrm{i}} = \mathrm{e}^{x\alpha+y\beta}$ and $u^{\mathrm{r}} = \mathrm{e}^{x\alpha-y\beta}$ for the TE polarization, where $\alpha = \mathrm{i}k\cos(\theta^{\mathrm{inc}})$, $\beta = \mathrm{i}k\sin(\theta^{\mathrm{inc}})$, θ^{inc} is incident angle; $k = 2\pi$; ϵ_{r} equals 1 for free space and $4 - i$ for filled media.

The characteristics of the TM and TE fields are shown in Figs. 2 and 3 for normal incidence. In both cases, we observe a perfect symmetry. Fig. 4 shows the RCS for the TM case. We note that having the exact solution in the exterior domain enables the efficient computation of the RCS using the analytical formulation,

$$\sigma(\psi) = \frac{16}{k\pi^2}\Psi^2(\psi),$$

where

$$\Psi(\psi) = \left|\sum_{n=1}^{\infty} \frac{\exp(\mathrm{i}n\pi/2)\sin(n\psi)}{J_n(kR) - \mathrm{i}Y_n(kR)} \int_0^{\pi} g(\theta)\sin(n\theta)\,\mathrm{d}\theta\right|.$$

In Fig. 5, we observe a smooth linkage between the numerical solutions in the interior domain and the analytical solutions in the exterior, indicating the reliability of our method in this aspect.

The relative errors in the L^2-norm and the H^1-norm are defined as

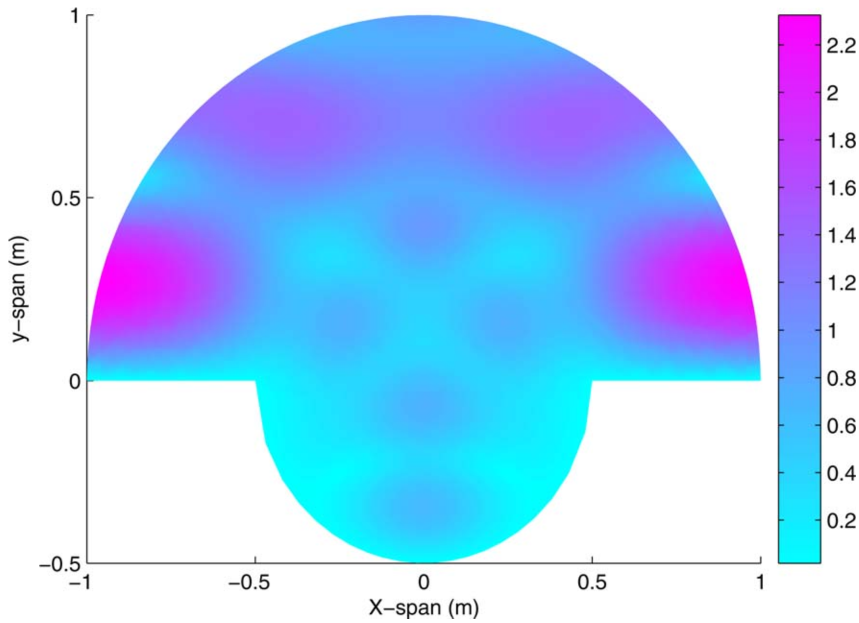

Fig. 2. Contour of magnitude of total field of TM polarization in interior domain, $\theta^{\mathrm{inc}} = \pi/2$.

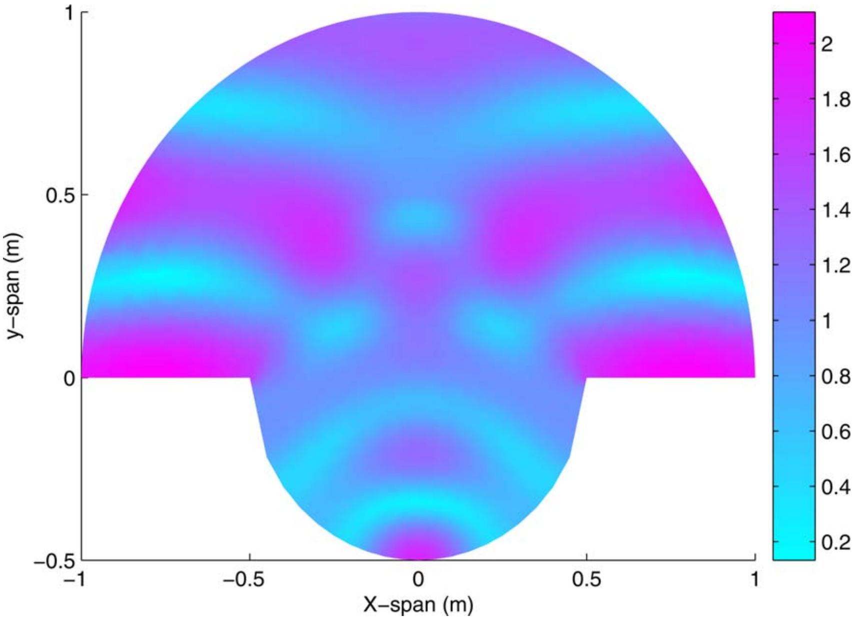

Fig. 3. Contour of magnitude of total field for TE polarization in interior domain, $\theta^{\text{inc}} = \pi/2$.

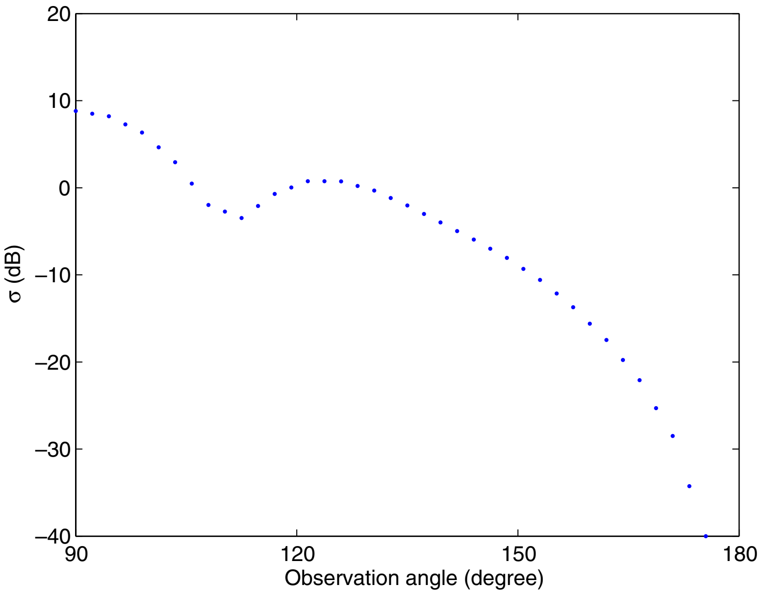

Fig. 4. RCS of the overfilled cavity for TM polarization.

$$
\text{error}_n = \begin{cases} \log_2 \frac{\|u_{A_n^e} - u_{A_{n-1}^e}\|_0}{\|u_{A_n^e}\|_0}, & \text{for } L^2\text{-norm}, \\ \log_2 \frac{\|u_{A_n^e} - u_{A_{n-1}^e}\|_1}{\|u_{A_n^e}\|_1}, & \text{for } H^1\text{-norm}, \end{cases}
$$

where A_n^e is the average element area at the nth mesh refinement. The relative errors are plotted against the reciprocal of the average element area $1/A_n^e$ in Fig. 6. As expected, we observe that the error in the H^1-norm is of order $\mathcal{O}(h)$, versus $\mathcal{O}(h^2)$ of the L^2-norm, where h is the element dimension.

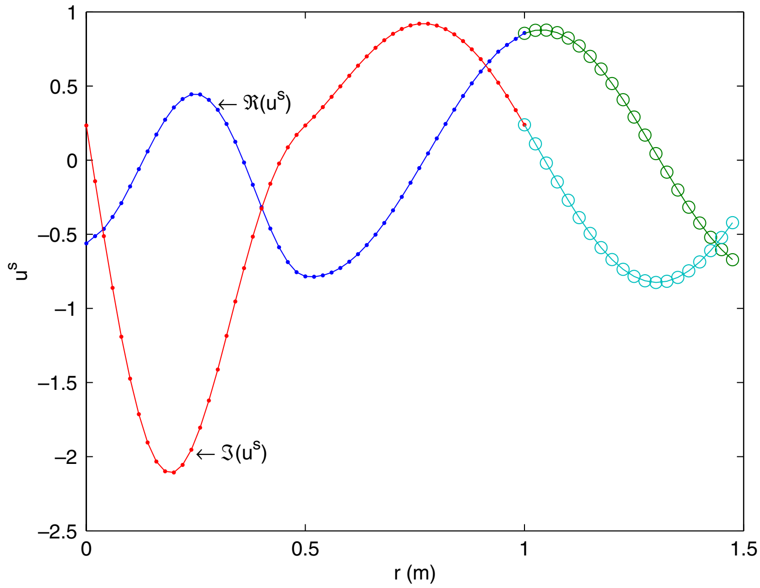

Fig. 5. Linkage of scattering field on $\mathscr{B}_R$ for TM polarization, $\theta = \theta^{\text{inc}} = \pi/2$.

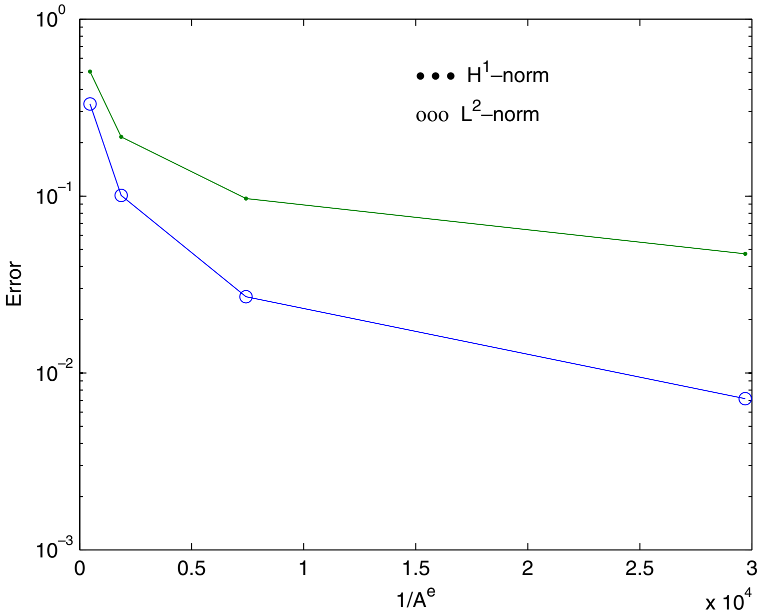

Fig. 6. Relative error versus the reciprocal of average element area for TE polarization, $\theta^{\text{inc}} = \pi/2$.

6. Conclusion

We have presented a finite element/Fourier series method for analyzing the scattering from inhomogeneous overfilled cavities embedded in the infinite ground plane. Our results indicate that the scattering problem in both TM and TE polarizations attains a unique weak solution for a general cavity medium. Our numerical experiments further demonstrate the realizability and efficiency of the method. We believe this is the first rigorous mathematical treatment supported by accurate numerical results of the important problem of electromagnetic scattering by overfilled cavities, and the approach can be generalized to three-dimensional scattering problems involving protruding cavities.

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